IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 05, Issue 03 (March. 2015), ||V1|| PP 33-47

BMAP/M/C Bulk Service Queue with Randomly Varying Environment

Rama.G, Ramshankar.R, Sandhya.R^3, Sundar.V^4, Ramanarayanan.R
Independent Researcher B. Tech, Vellore Institute of Technology, Vellore, India
Independent Researcher MS9EC0, University of Massachusetts, Amherst, MA, USA
Independent Researcher MSPM, School of Business, George Washington University, Washington .D.C, USA
Senior Testing Engineer, ANSYS Inc., 2600, Drive, Canonsburg, PA 15317, USA
Professor of Mathematics, (Retired), Vel Tech Dr. RR & Dr.SR Technical University, Chennai.

Abstract: This paper studies two stochastic BMAP arrival and bulk service C server queues (A) and (B) with k varying environments. The arrivals to the queue are governed by a batch Markovian arrival process of i version and the bulk service times are exponential with parameter μ_i in the environment i for $1 \le i \le k^*$ respectively. When the environment changes from i to j, changes occur for arrival and service as follows: the arrival BMAP representation changes from the i version to the j version, the residual arrival time starts with the stationary probability vector of the j version BMAP, it becomes the initial j version upon arrival of customers and the exponential service time parameter changes from μ_i to μ_j for $1 \le i, j \le k^*$. The system has infinite storing capacity and the service bulk sizes are finite valued random variables. Matrix partitioning method is used to study the models. In Model (A) the maximum of the arrival sizes M in all the environments is greater than the maximum of the service sizes N in all the environments, (M > N), and the infinitesimal generator is partitioned as blocks of the sum of the number of BMAP phases of all environments times the maximum of the arrival sizes for analysis. In Model (B) the maximum of the arrival sizes M in all the environments is less than the maximum of the service sizes N in all the environments, (M < N), where the infinitesimal generator is partitioned using blocks of the sum of the number of BMAP phases of all environment times the maximum of the service sizes for analysis. Five different cases associated with C, M and N due to partitions are treated. They are namely, (A1) M >N ≥ C, (A2) M ≥ C >N (A3) C >M >N, which come up in Model (A); (B1) N ≥ C and (B2) C >N, which come up in Model (B) respectively. For the cases when C ≤ M or N Matrix Geometric results are obtained and for the cases when C > both M and N Modified Matrix Geometric results are presented. The basic system generator is seen as a block circulant matrix in all the cases. The stationary queue length probabilities, its expected values, its variances and probabilities of empty queue levels are derived for the models using Matrix Methods. Numerical examples are presented for illustration

Keywords**:** *Block Circulant, BMAP Arrival, Bulk Service, C servers, Infinitesimal Generator, Matrix methods.*

I. INTRODUCTION

In this paper two multi server queues with batch Markovian arrival process (BMAP) and bulk service have been studied with random environments using matrix geometric methods. For M/M/1 bulk queues with random environment models one may refer to Rama Ganesan, Ramshankar and Ramanarayanan [1] and M/M/C bulk queues with random environment models are of interest in Sandhya, Sundar, Rama, Ramshankar and Ramanarayanan [2]. PH/PH/1 bulk queues without variation of environments have been treated by Ramshankar, Rama Ganesan and Ramanarayanan [3] and the same type of queues with random variation of environments are studied by Ramshankar, Rama, Sandhya, Sundar and Ramanarayanan [4]. Bini, Latouche and Meini [5] have studied numerical methods for Markov chains. Chakravarthy and Neuts [6] have discussed in depth a multi-server queue model. Gaver, Jacobs and Latouche [7] have treated birth and death models with random environment. Latouche and Ramaswami [8] have analyzed Analytic methods. For matrix geometric methods and models one may refer Neuts [9]. Batch Markovian arrival processes are presented by Lucantony [10] and are analyzed also by Cordeiro and Khroufch [11]. The models considered in this paper are general compared to existing queue models in literature since a BMAP arrival and multi server bulk service queue with random environment has not been studied at any depth so far. The number of servers increases with the arrival of number of customers till it becomes C. Usually the partitions of the bulk arrival models have M/G/1 upper-Heisenberg block matrix structure with zeros below the first sub diagonal. The decomposition of a Toeplitz sub matrix of the infinitesimal generator is required to find the stationary probability vector. In this paper the partitioning of the matrix is carried out in a way that the stationary probability vectors have a Matrix Geometric solution or a Modified Matrix Geometric solution for infinite capacity C server bulk arrival and bulk service queues with randomly varying environments.

Two models (A) and (B) on BMAP/M/C bulk queue systems under k* varying environments with infinite storage space for customers are studied here using the block partitioning method. The M/PH/1 and PH/M/C queues with random environments have been studied by Usha in [12] and [13] without bulk arrivals and bulk services. It has been noticed by Usha in [12, 13] that when the environment changes the remaining arrival and service times are to be completed in the new environment. The residual arrival time and the residual service time distributions in the new environment are to be considered at an arbitrary epoch since the spent arrival time and the spent service time have been in the previous environment with distinct sizes of PH phase. Further new arrival time and new service time from the start using initial PH distributions of the new environment cannot be considered since the arrival and the service have been partly completed in the previous environment indicating the stationary versions of the arrival and service distributions in the new environments are to be used for the completions of the residual arrival and service times in the new environment and on completion of the same the next arrival and service onwards they have initial versions of the PH distributions of the new environment. The stationary version of the distribution for residual time has been well explained in Qi-Ming He [14] where it is named as equilibrium PH distribution. Ramshankar, Rama, Sandhya, Sundar, Ramanarayanan in [4] have studied PH/PH/1 queue models with bulk arrival, bulk service with random environment introducing the stationary version for the residual times. In this paper the stationary probability starting vector of the new version is used when the environment changes for the residual arrival time and it becomes the initial new version of BMAP distribution after the arrival. Model (A) presents the case when M, the maximum of all the maximum arrival sizes in the environments is bigger than N, the maximum of all the maximum service sizes in all the environments. In Model (B), its dual, N is bigger than M, is treated. In general in Queue models, the state space of the system has the first co-ordinate indicating the number of customers in the system but here the customers in the system are grouped and considered as members of M sized blocks of customers (when M >N) or N sized blocks of customers (when N > M) for finding the rate matrix. For the C server system under consideration, Model (A) gives three cases namely (A1) M > N ≥ C, (A2) M ≥ C > N and (A3) C > M > N and Model (B) gives two cases namely (B1) N ≥ C, and (B2) C > N. The case M=N with various C values can be treated using Model (A) or Model (B). The matrices appearing as the basic system generators in these models due to block partitions are seen as block circulant matrices. The stationary probability of the number of customers waiting for service, the expected queue length, the variance and the probability of empty queue are derived for these models. Numerical cases are presented to illustrate their applications. The paper is organized in the following manner. In section II and section III the BMAP/M/C bulk service queues with randomly varying environment in which maximum arrival size M is greater than maximum service size N and the maximum arrival size M less than the maximum service size N are studied respectively with their sub cases. In section IV numerical cases are presented.

II. MODEL (A). MAXIMUM ARRIVAL SIZE M GREATER THAN MAXIMUM SERVICE SIZE N

2.1Assumptions for M > N.

(i) There are k* environments. The environment changes as per changes in a continuous time Markov chain with infinitesimal generator Q_1 of order k* with stationary probability vector ϕ.

(ii) In the environment i for 1 ≤ i ≤ k*, the batch arrivals occur in accordance with Batch Markovian Arrival Process with matrix representation for the rates of batches of size m ≤ M_i given by the finite sequence $\{D_m^i, 0 \leq m \leq M_i\}$ with phase order k_i where D_0^i has negative diagonal elements and its other elements are non-negative; D_m^i has non-negative elements for 1 ≤ m ≤ M_i. Let $D^i = \sum_{m=0}^{M} D_m^i$ and φ_i be the stationary probability vector of the generator matrix D^i with $\varphi_i D^i = 0$ and φ_ie = 1 for 1 ≤ i ≤ k*.

(iii) When the environment changes from i to j for 1 ≤ i, j ≤ k*, the arrival process BMAP of the j version starts as per stationary (equilibrium) probability vector of the j version of the arrival process for the completion of the residual arrival time there after the arrivals occur as per BMAP of the j version, namely, { D_m^j 0 ≤ m ≤ M_j }.

(iv)Customers are served in batches of different bulk sizes. There are s servers to serve when s customers are present in the system for 1≤ s ≤ C. When C or more than C customers are present in the system the number of servers to serve customers is C. In the environment i for1 ≤ i ≤ k*, the time between consecutive bulk services has exponential distribution with parameter sμ_i when s customers (s servers) are in the system for 1≤ s ≤ C and with parameter Cμ_i when C or more than C customers (C servers)are present where μ_i is the parameter of single server exponential service time distribution. At each service epoch in the environment i, ψ_i customers are served with probabilities given by P (ψ_i = j) = q_j^i for 1≤ j ≤ N_i when more than N_i customers are waiting for service where $\sum_{j=1}^{N_i} q_j^i$ =1. When n customers n < N_i are in the system, then j customers are served with probability, q_j^i for 1≤ j ≤ n-1 and n customers are served with probability $\sum_{j=n}^{N_i} q_j^i$ for 1 ≤ i ≤ k*.

(v) When the environment changes from i to j, the exponential service time parameter of single server changes from μ_i *to* μ_j , the bulk service size ψ_i changes to ψ_j and the maximum service size N_i changes to N_j.

(vi) The maximum batch arrival size of all BMAPs', M= $\max_{1\leq i\leq k*} M_i$ is greater than the maximum service size N= $\max_{1\leq i\leq k*} N_i$

2.2.Analysis

There are three sub cases for this model namely (A1) M > N ≥ C, (A2) M ≥ C >N and (A3) C > M >N. Sub Cases (A1) and (A2) admit Matrix Geometric solutions and they are treated in sub section (2.2.1). Modified Matrix Geometric solution is presented for Sub Case (A3) which is studied in sub section (2.2.2). The state of the system of the continuous time Markov chain X (t) under consideration is presented as follows. X (t) = {(n, m, i, j): for 0 ≤ m ≤ M-1; 1 ≤ i ≤ k*, 1 ≤ j ≤ k_i and n ≥ 0}(1) The chain is in the state (n, m, i, j) when the number of customers in the system is n M + m, for 0 ≤ m ≤ M-1, 0 ≤ n < ∞, the environment is i for 1 ≤ i ≤ k* and the arrival phase is j for 1 ≤ j ≤ k_i. When the number of customers in the system is r, then r is identified with (n, m) where r on division by M gives n as the quotient and m as the remainder. . Let the survivor probabilities of services ψ_i be respectively for the environment state i for 1 ≤ i ≤ k*. P(ψ_i>m)= Q_m^i=1-$\sum_{n=1}^{m} q_n^i$, for 1 ≤ m ≤ N_i -1 (2)

Q_m^i=0 for m ≥ N_i and $Q_0^i = 1$. (3)

2.2.1 Sub Cases: (A1) M > N ≥ C and (A2) M ≥ C > N

When M > N ≥ C or M ≥ C > N, the BMAP/M/C bulk queue admits matrix geometric solution as follows. The chain X (t) describing them, has the infinitesimal generator $Q_{A,2.1}$ of infinite order which can be presented in block partitioned form given below.

$$Q_{A,2.1} = \begin{bmatrix} B_1 & A_0 & 0 & 0 & . & . & . & \cdots \\ A_2 & A_1 & A_0 & 0 & . & . & . & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & . & . & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (4)$$

In (4) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots.\underline{n}$, ... Here the vector $\underline{n}$ is of type 1 x M $\sum_{i=1}^{k*} k_i$ and $\underline{n}$ = ((n, 0, 1, 1),(n, 0, 1, 2)…(n, 0, 1, k_1), (n, 0, 2, 1),(n, 0, 2, 2)…(n, 0, 2, k_2),…,(n, 0, k*, 1),(n, 0, k*, 2)…(n, 0, k*, k_{k*}), (n, 1, 1, 1),(n, 1, 1, 2)…(n, 1, 1, k_1), (n, 1, 2, 1),(n, 1, 2, 2)…(n, 1, 2, k_2),…,(n, 1, k*, 1),(n, 1, k*, 2)…(n, 1, k*, k_{k*}),…, (n, M-1, 1, 1),(n, M-1, 1, 2)…(n, M-1, 1, k_1), (n, M-1, 2, 1),(n, M-1, 2, 2)…(n, M-1, 2, k_2),…,(n, M-1, k*, 1),(n, M-1, k*, 2)…(n, M-1, k*, k_{k*})) for n ≥ 0. The matrices B_1 *and* A_1 have negative diagonal elements, they are of order M$\sum_{i=1}^{k*} k_i$ and their off diagonal elements are non- negative. The matrices A_0, *and* A_2 have nonnegative elements and are of order M$\sum_{i=1}^{k*} k_i$ and they are given below.

Let $Q_i^{'}$= $D_0^i + (-C\mu_i + (Q_1)_{i,i})I_{k_i}$ for 1 ≤ i ≤ k* (5)

where I indicates the identity matrix of order given in the suffix, $Q_i^{'}$ is of order k_i. Considering the change of environment switches on stationary version of BMAP arrival in the new environment, the following matrix Ω of order $\sum_{i=1}^{k*} k_i$ is defined which is concerned with change of environment during arrival time and service time.

$$\Omega = \begin{bmatrix} Q'_1 & \Omega_{1,2} & \Omega_{1,3} & \cdots & \Omega_{1,k*} \\ \Omega_{2,1} & Q'_2 & \Omega_{2,3} & \cdots & \Omega_{2,k*} \\ \Omega_{3,1} & \Omega_{3,2} & Q'_3 & \cdots & \Omega_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{k*,1} & \Omega_{k*,2} & \Omega_{k*,3} & \cdots & Q'_{k*} \end{bmatrix} \quad (6)$$

where $\Omega_{i,j}$ is a rectangular matrix of type k_ix k_j whose all rows are equal to $(Q_1)_{i,j}\ \varphi_j$ for i ≠ j , 1 ≤ i, j ≤ k*. In the environment i, for 1 ≤ i ≤ k*, the matrix of arrival rates of n customers corresponding to the arrival in BMAP is D_n^i which is a matrix with non-negative elements for 1 ≤ n ≤ M_i and D_n^i = 0 matrix for n > M_i (7) and the rate with which n customers are served by a single server for 1≤ n ≤ N_i is given by $S_{i,n}^{'}$=$\mu_i q_n^i$ and $S_{i,n}^{'}$= 0 if n > N_i. (8)

Let $\Lambda_n = \begin{bmatrix} D_n^1 & 0 & 0 & \cdots & 0 \\ 0 & D_n^2 & 0 & \cdots & 0 \\ 0 & 0 & D_n^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_n^{k*} \end{bmatrix}$ for $1 \leq n \leq M$ (9)

In (9) Λ_n is a square matrix of order $\sum_{i=1}^{k*} k_i$; D_n^j is a square matrix of order k_j for $1 \leq j \leq k^*$ and 0 appearing as (i,j) component of (9) is a block zero rectangular matrix of type $k_i \times k_j$.

Let $U_n = \begin{bmatrix} S'_{1,n} I_{k_1} & 0 & 0 & \cdots & 0 \\ 0 & S'_{2,n} I_{k_2} & 0 & \cdots & 0 \\ 0 & 0 & S'_{3,n} I_{k_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S'_{k*,n} I_{k_{k*}} \end{bmatrix}$ for $1 \leq n \leq N$ (10)

In (10) U_n is a square matrix of order $\sum_{i=1}^{k*} k_i$; $S'_{j,n} I_{k_j}$ is a square matrix of order k_j for $1 \leq j \leq k^*$ and 0 appearing as (i, j) component of (10) is a block zero rectangular matrix of type $k_i \times k_j$. The matrix A_i for i = 0, 1, 2 are as follows.

$$A_0 = \begin{bmatrix} \Lambda_M & 0 & \cdots & 0 & 0 & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 \\ \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 & 0 \\ \Lambda_{M-3} & \Lambda_{M-2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M & 0 & 0 \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M \end{bmatrix} \quad (11)$$

$$A_2 = \begin{bmatrix} 0 & \cdots & 0 & CU_N & CU_{N-1} & \cdots & CU_2 & CU_1 \\ 0 & \cdots & 0 & 0 & CU_N & \cdots & CU_3 & CU_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & CU_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (12)$$

$$A_1 = \begin{bmatrix} \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ CU_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ CU_2 & CU_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_N & CU_{N-1} & CU_{N-2} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & CU_N & CU_{N-1} & \cdots & CU_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & CU_N & \cdots & CU_2 & CU_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} & CU_{N-2} & \cdots & \Omega & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & CU_N & CU_{N-1} & \cdots & CU_1 & \Omega \end{bmatrix} \quad (13)$$

For defining the matrices B_1 the following component matrices are required Using (2) and (3) let $V'_{i,n} = \mu_i Q_n^i I_{k_i}$ for $1 \leq n \leq N-1$ which is a matrix of order k_i for $1 \leq i \leq k^*$and let

$V_n = \begin{bmatrix} V'_{1,n} & 0 & 0 & \cdots & 0 \\ 0 & V'_{2,n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & V'_{k*,n} \end{bmatrix}$ for $1 \leq n \leq N-1$. (14)

This is a matrix of order $\left(\sum_{i=1}^{k*} k_i\right)$ and 0 appearing in the (i, j) component is a 0 matrix of type $k_i \times k_j$ for $1 \leq i, j \leq k^*$.

Let U = $\begin{bmatrix} \mu_1 I_{k_1} & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 I_{k_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{k*} I_{k_1} \end{bmatrix}$ (15)

In (15), U is matrix of order $\left(\sum_{i=1}^{k*} k_i\right)$ and 0 appearing in the (i, j) component is a rectangular 0 matrix of type k_i x k_j for 1 ≤ i, j ≤ k*. To write B_1 the block for $\underline{0}$ is to be considered which has queue length L= 0, 1, 2…M-1. When L = 0 there is only arrival process without service. The change in the environment from i to j switches on BMAP j version as per stationary (equilibrium) probability vector in the new environment j whenever it occurs for 1 ≤ i, j, ≤ k*. In the empty queue (L=0) when an arrival occurs in the environment i both the arrival time and the service time start. In block $\underline{0}$ when L =1, 2,…,M-1 all the processes arrival, service and environment are active as in other blocks $\underline{n}$ for n > 0. Considering the change of environment switches on BMAP arrival process in the new environment through the stationary (equilibrium) probability vector when the queue is empty, the following matrix Ω' of order $\sum_{i=1}^{k*} k_i$ is defined which is concerned with the change of environment during arrival time and is similar to Ω defined in (6).

$$\Omega' = \begin{bmatrix} T'_1 & \Omega_{1,2} & \Omega_{1,3} & \cdots & \Omega_{1,k*} \\ \Omega_{2,1} & T'_2 & \Omega_{2,3} & \cdots & \Omega_{2,k*} \\ \Omega_{3,1} & \Omega_{3,2} & T'_3 & \cdots & \Omega_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{k*,1} & \Omega_{k*,2} & \Omega_{k*,3} & \cdots & T'_{k*} \end{bmatrix} \tag{16}$$

Here $T'_i = D_0^i + diag(Q_1)_{i,i}$ and $\Omega_{i,j}$ is a rectangular matrix of type k_i x k_j whose all rows are equal to $(Q_1)_{i,j}$ φ_j presenting the rates of changing to phases in the new environment for i ≠ j and 1 ≤ i, j ≤ k*.

The matrix B_1 for Sub Case (A1) where N > C and Sub Case (A2) where C > N are given below in (17) and (18) respectively. For the case when C=N, the matrix B_1 may be written by placing C in place of N in the N-th block row in (18) and there after the multiplier of U_j is C. Let $Q'_{1,j} = \Omega' - jU$ for 0 ≤ j ≤ C and $Q'_{1,C} = \Omega$

$$B_1 = \begin{bmatrix} Q'_{1,0} & A_1 & A_2 & A_3 & \cdots & A_C & A_{C+1} & A_{C+2} & \cdots & A_N & \cdot & \cdots & A_{M-N-2} & A_{M-N-1} & \cdots & A_{M-2} & A_{M-1} \\ U & Q'_{1,1} & A_1 & A_2 & \cdots & A_{C-1} & A_C & A_{C+1} & \cdots & A_{N-1} & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-3} & A_{M-2} \\ 2V_1 & 2U_1 & Q'_{1,2} & A_1 & \cdots & A_{C-2} & A_{C-1} & A_C & \cdots & A_{N-2} & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-4} & A_{M-3} \\ 3V_2 & 3U_2 & 3U_1 & Q'_{1,3} & \cdots & A_{C-3} & A_{C-2} & A_{C-1} & \cdots & A_{N-3} & \cdot & \cdots & \cdot & \cdot & \ddots & A_{M-5} & A_{M-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CV_{C-1} & CU_{C-1} & CU_{C-2} & CU_{C-3} & \cdots & Q'_{1,C} & A_1 & A_2 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-C-2} & A_{M-C-1} \\ CV_C & CU_C & CU_{C-1} & CU_{C-2} & \cdots & CU_1 & \Omega & A_1 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-C-3} & A_{M-C-2} \\ CV_{C+1} & CU_{C+1} & CU_C & CU_{C-1} & \cdots & CU_2 & CU_1 & \Omega & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-C-4} & A_{M-C-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CV_{N-1} & CU_{N-1} & CU_{N-2} & CU_{N-3} & \cdots & CU_{N-C} & CU_{N-C-1} & CU_{N-C-2} & \cdots & \Omega & A_1 & \cdots & A_{M-2N-2} & A_{M-2N-1} & \cdots & A_{M-N-2} & A_{M-N-1} \\ 0 & CU_N & CU_{N-1} & CU_{N-2} & \cdots & CU_{N-C+1} & CU_{N-C} & CU_{N-C-1} & \cdots & CU_1 & \Omega & \cdots & \cdot & \cdot & \cdots & A_{M-N-3} & A_{M-N-2} \\ 0 & 0 & CU_N & CU_{N-1} & \cdots & CU_{N-C+2} & CU_{N-C+1} & CU_{N-C} & \cdots & CU_2 & CU_1 & \cdots & \cdot & \cdot & \cdots & A_{M-N-4} & A_{M-N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & \Omega & A_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & \Omega \end{bmatrix} \tag{17}$$

$$B_1 = \begin{bmatrix} Q'_{1,0} & A_1 & A_2 & A_3 & \cdots & A_N & A_{N+1} & \cdots & A_C & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-2} & A_{M-1} \\ U & Q'_{1,1} & A_1 & A_2 & \cdots & A_{N-1} & A_N & \cdots & A_{C-1} & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-3} & A_{M-2} \\ 2V_1 & 2U_1 & Q'_{1,2} & A_1 & \cdots & A_{N-2} & A_{N-1} & \cdots & A_{C-2} & \cdot & \cdots & \cdot & \cdot & \cdots & A_{M-4} & A_{M-3} \\ 3V_2 & 3U_2 & 3U_1 & Q'_{1,3} & \cdots & A_{N-3} & A_{N-2} & \cdots & A_{C-3} & \cdot & \cdots & \cdot & \cdot & \ddots & A_{M-5} & A_{M-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ NV_{N-1} & NU_{N-1} & NU_{N-2} & NU_{N-3} & \cdots & Q'_{1,N} & A_1 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & (N+1)U_N & (N+1)U_{N-1} & (N+1)U_{N-2} & \cdots & (N+1)U_1 & Q'_{1,N+1} & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & (N+2)U_N & (N+2)U_{N-1} & \cdots & (N+2)U_2 & (N+2)U_1 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & Q'_{1,C} & A_1 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & \Omega & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & \Omega & A_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & \Omega \end{bmatrix} \tag{18}$$

For the case when M = C, the multiplier C does not appear as a multiplier for the U_j matrices in the matrix B_1 in (18) in the $\underline{0}$ block of (4) and C appears as a multiplier for all U_j matrices in the matrices of A_1 and A_2 from row block $\underline{1}$ onwards. The basic generator of the bulk queue which is concerned with only the arrival and service is a matrix of order $[M \sum_{i=1}^{k*} k_i]$ given below in (21) where $Q''_A = A_0 + A_1 + A_2$ (19)

Its probability vector w' gives, $w'Q''_A = 0$ and w'. e = 1 (20)

It is well known that a square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a circulant matrix. It is very interesting to note that the matrix Q''_A is a block circulant matrix where each block matrix is rotated one block to the right relative to the preceding block partition.

$Q''_A =$

$$\begin{bmatrix} \Omega+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N & \cdots & \Lambda_{M-2}+CU_2 & \Lambda_{M-1}+CU_1 \\ \Lambda_{M-1}+CU_1 & \Omega+\Lambda_M & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3}+CU_3 & \Lambda_{M-2}+CU_2 \\ \Lambda_{M-2}+CU_2 & \Lambda_{M-1}+CU_1 & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4}+CU_3 & \Lambda_{M-3}+CU_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_{M-N+2}+CU_{N-2} & . & \cdots & . & . & . & \cdots & \Lambda_{M-N}+CU_N & \Lambda_{M-N+1}+CU_{N-1} \\ \Lambda_{M-N+1}+CU_{N-1} & . & \cdots & . & . & . & . & \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N \\ \Lambda_{M-N}+CU_N & . & \cdots & \Omega+\Lambda_M & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N & \cdots & \Lambda_{M-1}+CU_1 & \Omega+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2}+CU_2 & \Lambda_{M-1}+CU_1 & \Omega+\Lambda_M & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-N}+CU_N & \Lambda_{M-N+1}+CU_{N-1} & \Lambda_{M-N+2}+CU_{N-2} & \cdots & \Omega+\Lambda_M & \Lambda_1 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-1} & \Lambda_{M-N}+CU_N & \Lambda_{M-N+1}+CU_{N-1} & \cdots & \Lambda_{M-1}+CU_1 & \Omega+\Lambda_M \end{bmatrix} \quad (21)$$

In (21), the first block-row of type $[\sum_{i=1}^{k*} k_i]$ x$[M\sum_{i=1}^{k*} k_i]$ is, $W = (\Omega+\Lambda_M, \Lambda_1, \Lambda_2, \ldots, \Lambda_{M-N-2}, \Lambda_{M-N-1}, \Lambda_{M-N}+CU_N, \ldots, \Lambda_{M-2}+CU_2, \Lambda_{M-1}+CU_1)$ which gives as the sum of the blocks $(\Omega+\Lambda_M) + \Lambda_1 + \Lambda_2 + \ldots + \Lambda_{M-N-2} + \Lambda_{M-N-1} + \Lambda_{M-N} + CU_N + \ldots + \Lambda_{M-2} + CU_2 + \Lambda_{M-1} + CU_1 = \Omega''$ which is the matrix given by

$$\Omega'' = \begin{bmatrix} Q''_1 & \Omega_{1,2} & \Omega_{1,3} & \cdots & \Omega_{1,k*} \\ \Omega_{2,1} & Q''_2 & \Omega_{2,3} & \cdots & \Omega_{2,k*} \\ \Omega_{3,1} & \Omega_{3,2} & Q''_3 & \cdots & \Omega_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{k*,1} & \Omega_{k*,2} & \Omega_{k*,3} & \cdots & Q''_{k*} \end{bmatrix} \quad (22)$$

where using (5) and (6), $Q''_i = D^i +$ diag $(Q_1)_{i,i}$ for $1 \le i \le k*$. The stationary probability vector of the basic generator given in (21) is required to get the stability condition. Consider the vector

$w = (\phi_1\varphi_1, \phi_2\varphi_2, \ldots, \phi_{k*}\varphi_{k*})$ (23)

where $\phi = (\phi_1, \phi_2, \ldots, \phi_{k*})$ is the stationary probability vector of the environment, $\varphi_i = (\varphi_{i,j})$ is the stationary probability vector of the arrival BMAP D^i for $1 \le i \le k*$. It may be noted $\phi_i\varphi_i D^i = 0$. This gives $\phi_i\varphi_i Q''_i = \phi_i (Q_1)_{i,i}\, \varphi_i$ for $1 \le i \le k*$. Now the first column of the matrix multiplication of wΩ'' is $\phi_1 (Q_1)_{1,1}\varphi_{1,1} + \phi_2 (Q_1)_{2,1}\varphi_{11}[\varphi_2 e] + \ldots + \phi_{k*}(Q_1)_{k*,1}\varphi_{11}[\varphi_{k*} e] = 0$ since $(\varphi_i)e = 1$ and $\phi Q_1 = 0$. In a similar manner it can be seen that the first column block of the matrix multiplication of wΩ'' is $\phi_1 (Q_1)_{1,1}\varphi_1 + \phi_2 (Q_1)_{2,1}\varphi_1[(\varphi_2)e] + \ldots + \phi_{k*}(Q_1)_{k*,1}\varphi_1[(\varphi_{k*})e] = 0$ and i-th column block is $\phi_1 (Q_1)_{1,i}\varphi_i[(\varphi_1)e] + \phi_2 (Q_1)_{2,i}\varphi_i[(\varphi_2)e] + \ldots + \phi_i (Q_1)_{i,i}\varphi_i + \ldots + \phi_{k*}(Q_1)_{k*,i}\varphi_i[(\varphi_{k*})e] = 0$. This shows that $w(\Omega+\Lambda_M) + w\Lambda_1 + w\Lambda_2 + \ldots + w\Lambda_{M-N-2} + w\Lambda_{M-N-1} + w\Lambda_{M-N} + wCU_N + \ldots + w\Lambda_{M-2} + wCU_2 + w\Lambda_{M-1} + wCU_1 =$ w Ω''=0. So (w, w,…,w) .W= 0 = (w, w, ….w) W' where W' is the transpose W. This shows (w, w...w) is the left eigen vector of Q'_A and the corresponding probability vector is

$$w' = \left(\frac{w}{M}, \frac{w}{M}, \frac{w}{M}, \ldots\ldots, \frac{w}{M}\right) \quad (24)$$

where w is given by (23). Neuts [5], gives the stability condition as, $w' A_0\, e < w' A_2\, e$ where w' is given by (24). Taking the sum cross diagonally in the A_0 *and* A_2 matrices, it can be seen using (9), (10), (11) and (12) that

$w' A_0\, e = \frac{1}{M} w'(\sum_{n=1}^{M} n\Lambda_n)e = \frac{1}{M} \left(\sum_{n=1}^{M}\sum_{i=1}^{k*} n\,\phi_i(\varphi_i D_n^i)e\right) = \frac{1}{M}(\sum_{i=1}^{k*}\phi_i \sum_{n=1}^{M} n\,(\varphi_i D_n^i)e$
$= \frac{1}{M}(\sum_{i=1}^{k*}\phi_i\varphi_i(\sum_{n=1}^{M} n\, D_n^i)e < w' A_2\, e = \frac{1}{M} w(\sum_{n=1}^{N} nCU_n)e = \frac{C}{M}\left(\sum_{n=1}^{N}\sum_{i=1}^{k*} n\,\phi_i\varphi_i\mu_i q_n^i e\right) =$
$\frac{C}{M}\left(\sum_{i=1}^{k*}\phi_i\mu_i \sum_{n=1}^{N} n\, q_n^i\right) = \frac{C}{M}(\sum_{i=1}^{k*}\phi_i\mu_i \mathrm{E}(\psi_i)$. This gives the stability condition as

$$\sum_{i=1}^{k*}\phi_i\varphi_i(\sum_{n=1}^{M} n\, D_n^i)e < C\sum_{i=1}^{k*}\phi_i\mu_i \mathrm{E}(\psi_i) \quad (25)$$

This is the stability condition for the BMAP/M/C bulk service queue with random environment for Sub Case (A1) M > N ≥ C and Sub Case (A2) M ≥ C > N. When (25) is satisfied, the stationary distribution exists as proved in Neuts [9]. Let π (n, m, i, j), for 0 ≤ m ≤ M-1, 1 ≤ i ≤ k*, 1 ≤ j ≤ k_i and 0 ≤ n < ∞ be the stationary probability of the states in (1) and π_n be the vector of type 1xM$\sum_{i=1}^{k*} k_i$ with π_n = (π(n, 0, 1, 1), π(n, 0, 1, 2) … π(n, 0, 1, k_1), π(n, 0, 2, 1), π(n, 0, 2, 2),…π(n, 0,2, k_2)… π(n, 0, k*, 1), π(n, 0, k*, 2),…π(n, 0,k*, k_{k*})…… π(n, M-1, 1, 1), π(n, M-1, 1, 2) … π(n, M-1, 1, k_1), π(n, M-1, 2, 1), π(n, M-1, 2, 2),…π(n, M-1,2, k_2)… π(n, M-1, k*, 1), π(n, M-1, k*, 2),…π(n, M-1,k*, k_{k*})for n ≥ 0. The stationary probability vector $\pi = (\pi_0, \pi_1, \pi_3, \ldots\ldots)$ satisfies $\pi Q_{A,2.1}$=0 and πe=1. (26)

From (26), it can be seen $\pi_0 B_1 + \pi_1 A_2$=0. (27)

$\pi_{n-1}A_0 + \pi_n A_1 + \pi_{n+1}A_2 = 0$, for n ≥ 1. (28)

Introducing the rate matrix R as the minimal non-negative solution of the non-linear matrix equation A_0+RA_1+$R^2 A_2$=0, (29)

it can be proved (Neuts [9]) that π_n satisfies $\pi_n = \pi_0 R^n$ for n ≥ 1. (30)

Using (27) and (30), π_0 satisfies $\pi_0\, [B_1 + RA_2]$ =0 (31)

The vector π_0 can be calculated up to multiplicative constant by (31). From (26) and (30)

$\pi_0\ (I-R)^{-1}e$=1. (32)

Replacing the first column of the matrix multiplier of π_0 in equation (31) by the column vector multiplier of π_0 in (32), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is π_0 and this gives along with (30) all the stationary probabilities. The matrix R given in (29) is computed using recurrence relation $R(0)=0;\ R(n+1)=-A_0A_1^{-1}-R^2(n)A_2A_1^{-1}$, n ≥ 0. (33)

The iteration may be terminated to get a solution of R at an approximate level where $\|R(n+1)-R(n)\|$ < ε.

2.2.2 Sub Case: (A3) C > M > N

When C > M > N, the BMAP/M/C bulk queue admits a modified matrix geometric solution as follows. The chain X (t) describing this Sub Case (A3), can be defined as in (1) presented for Sub Cases (A1) and (A2). It has the infinitesimal generator $Q_{A,2.2}$ of infinite order which can be presented in block partitioned form given below. When C > M, let C = m* M + n* where m* is positive integer and n* is nonnegative integer with 0 ≤ n* ≤ M-1.

$$Q_{A,2.2}=\begin{bmatrix} B'_1 & A_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & A_{2,m*} & A_{1,m*} & A_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots \end{bmatrix} \qquad (34)$$

In (34) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots . \underline{n}, \ldots$. Here the vector $\underline{n}$ is of type 1xM$\sum_{i=1}^{k*}k_i$and $\underline{n}$ = ((n, 0, 1, 1),(n, 0, 1, 2)…(n, 0, 1, k_1), (n, 0, 2, 1),(n, 0, 2, 2)…(n, 0, 2, k_2),…,(n, 0, k*, 1),(n, 0, k*, 2)…(n, 0, k*, k_{k*}), (n, 1, 1, 1),(n, 1, 1, 2)…(n, 1, 1, k_1), (n, 1, 2, 1),(n, 1, 2, 2)…(n, 1, 2, k_2),…,(n, 1, k*, 1),(n, 1, k*, 2)…(n, 1, k*, k_{k*}),…, (n, M-1, 1, 1),(n, M-1, 1, 2)…(n, M-1, 1, k_1), (n, M-1, 2, 1),(n, M-1, 2, 2)…(n, M-1, 2, k_2),…,(n, M-1, k*, 1),(n, M-1, k*, 2)…(n, M-1, k*, k_{k*})) for n ≥ 0.The matrices $B'_1, A_{1,j}$ for $1 \le$ j $< m*\ and\ A_1$ have negative diagonal elements, they are of order M$\sum_{i=1}^{k*}k_i$and their off diagonal elements are non- negative. The matrices $A_0\ and A_2$ have nonnegative elements and are of order M$\sum_{i=1}^{k*}k_i$ and the matrices $A_0, A_1 and\ A_2$ are same as defined earlier for Sub Cases (A1) and (A2) in equations (11), (12) and (13). Since C > M the number of servers in the system s equals the number of customers in the system L up to customer length becomes C. Once number of customers becomes L ≥ C, the number of servers in the system remains C. When the number of customers becomes less than C, the number of servers reduces and equals the number of customers. The matrix $A_{2,j}$ for 1 ≤ j < m*-1 is given below.

$$A_{2,j}=\begin{bmatrix} 0 & \cdots & 0 & jMU_N & jMU_{N-1} & \cdots & jMU_2 & jMU_1 \\ 0 & \cdots & 0 & 0 & (jM+1)U_N & \cdots & (jM+1)U_3 & (jM+1)U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & (jM+N-2)U_N & (jM+N-2)U_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & (jM+N-1)U_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad (35)$$

The matrix $A_{2,m*}$ is as follows given in (36) when C = m*M + n* and n* is such that 0 ≤ n* ≤ N-1. Here the multiplier of U_j in the row block increases by one till the multiplier becomes C = m*M + n* and there after the multiplier is C for U_j for all blocks. When N ≤ n* ≤ M-1, $A_{2,m*}$ is same as in (35) for j = m*

$$A_{2,m*}=\begin{bmatrix} 0 & \cdots & 0 & (Mm*)U_N & (Mm*)U_{N-1} & \cdots & . & \cdots & (Mm*)U_2 & (Mm*)U_1 \\ 0 & \cdots & 0 & 0 & (Mm*+1)U_N & \cdots & . & \cdots & (Mm*+1)U_3 & (Mm*+1)U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & CU_N & \cdots & CU_{n*+2} & CU_{n*+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & CU_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad (36)$$

$$B'_1 = \begin{bmatrix} Q'_{1,0} & A_1 & A_2 & A_3 & \cdots & A_{N+1} & \cdots & . & . & \cdots & A_{M-2} & A_{M-1} \\ U & Q'_{1,1} & A_1 & A_2 & \cdots & A_N & \cdots & . & . & \cdots & A_{M-3} & A_{M-2} \\ 2V_1 & 2U_1 & Q'_{1,2} & A_1 & \cdots & A_{N-1} & \cdots & . & . & \cdots & A_{M-4} & A_{M-3} \\ 3V_2 & 3U_2 & 3U_1 & Q'_{1,3} & \cdots & A_{N-2} & \cdots & . & . & \ddots & A_{M-5} & A_{M-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ NV_{N-1} & NU_{N-1} & NU_{N-2} & NU_{N-3} & \cdots & A_1 & \cdots & . & . & \cdots & . & . \\ 0 & (N+1)U_N & (N+1)U_{N-1} & (N+1)U_{N-2} & \cdots & Q'_{1,N+1} & \cdots & . & . & \cdots & . & . \\ 0 & 0 & (N+2)U_N & (N+2)U_{N-1} & \cdots & (N+2)U_1 & \cdots & . & . & \cdots & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & (M-2)U_N & (M-2)U_{N-1} & \cdots & Q'_{1,M-2} & A_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & (M-1)U_N & \cdots & (M-1)U_1 & Q'_{1,M-1} \end{bmatrix} \quad (37)$$

In (37) as in previous sub case (6), (16), $Q'_{1,j} = \Omega' - jU$ for $0 \le j \le C$ and $Q'_{1,C} = \Omega$ are used. The matrix $A_{1,j}$ for $1 \le j \le m^*$-1 is as follows.

$$A_{1,j} = \begin{bmatrix} Q'_{1,jM} & A_1 & A_2 & \cdots & A_{M-N-2} & A_{M-N-1} & \cdots & A_{M-2} & A_{M-1} \\ (jM+1)U_1 & Q'_{1,jM+1} & A_1 & \cdots & A_{M-N-3} & A_{M-N-2} & \cdots & A_{M-3} & A_{M-2} \\ (jM+2)U_2 & (jM+2)U_1 & Q'_{1,jM+2} & \cdots & A_{M-N-4} & A_{M-N-3} & \cdots & A_{M-4} & A_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (jM+N)U_N & (jM+N)U_{N-1} & (jM+N)U_{N-2} & \cdots & Q'_{1,jM+N} & A_1 & \cdots & A_{M-N-2} & A_{M-N-1} \\ 0 & (jM+N+1)U_N & (jM+N+1)U_{N-1} & \cdots & (jM+N+1)U_1 & Q'_{1,jM+N+1} & \cdots & A_{M-N-3} & A_{M-N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (jM+M-2)U_N & (jM+M-2)U_{N-1} & \cdots & Q'_{1,jM+M-2} & A_1 \\ 0 & 0 & 0 & \cdots & 0 & (jM+M-1)U_N & \cdots & (jM+M-1)U_1 & Q'_{1,jM+M-1} \end{bmatrix} \quad (38)$$

The matrix $A_{1,m*}$ is as follows when C = m*M + n* and n* is such that 0 ≤ n* ≤ N-1. The multiplier of U_j increases by one till it becomes C = m*M + n* and thereafter in all the blocks the multiplier of U_j is C.

$$A_{1,m*} = \begin{bmatrix} Q'_{1,Mm*} & A_1 & A_2 & \cdots & A_{M-N-2} & A_{M-N-1} & \cdots & A_{M-2} & A_{M-1} \\ (Mm*+1)U_1 & Q'_{1,jm*+1} & A_1 & \cdots & A_{M-N-3} & A_{M-N-2} & \cdots & A_{M-3} & A_{M-2} \\ (Mm*+2)U_2 & (jm*+2)U_1 & Q'_{1,jm*+2} & \cdots & A_{M-N-4} & A_{M-N-3} & \cdots & A_{M-4} & A_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (Mm*+n*-1)U_{n*-1} & (jm*+n*-1)U_{n*-2} & . & \cdots & . & . & \cdots & . & . \\ CU_{n*} & CU_{n*-1} & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CU_N & CU_{N-1} & CU_{N-2} & \cdots & Q'_{1,C} & A_1 & \cdots & A_{M-N-2} & A_{M-N-1} \\ 0 & CU_N & CU_{N-1} & \cdots & CU_1 & Q'_{1,C} & \cdots & A_{M-N-3} & A_{M-N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} & \cdots & Q'_{1,C} & A_1 \\ 0 & 0 & 0 & \cdots & 0 & CU_N & \cdots & CU_1 & Q'_{1,C} \end{bmatrix} \quad (39)$$

When n* = N or n* > N then, in the matrix $A_{1,m*}$, there is slight change in the elements. When n* = N, in the N+1 block row and thereafter C appears as multiplier of U_j, and when n* > N with n* = N + r for 1 ≤ r ≤ M-N-1, in the n*+1 block row U_N appears in the r + 1 column block. C appears as multiplier for it and as the multiplier of U_j thereafter in all row blocks respectively. The basic system generator for this Sub Case is same as (21) with probability vector as given in (24). The stability condition is as presented in (25). Once the stability condition is satisfied the stationary probability vector exists by Neuts [9]. As in the previous Sub Cases, $\pi Q_{A,2.2}$=0 and πe=1. (40)

The following may be noted. $\pi_n A_0+\pi_{n+1}A_1+\pi_{n+2}A_2$ = 0, for n ≥ m*, the rate matrix R is same as in previous Sub Cases with same iterative method for solving the same and π_n satisfies $\pi_n = \pi_{m*} R^{n-m*}$ for n ≥ m*. (41)

The set of equations available from (40) are $\pi_0 B'_1+\pi_1 A_{2,1}$= 0, (42)

$\pi_i A_0+\pi_{i+1}A_{1,i+1}+\pi_{i+2}A_{2,i+2}$ = 0, for 0 ≤ i ≤ m*-2 (43)

and $\pi_{m*-1}A_0+\pi_{m*}A_{1,m*}+\pi_{m*+1}A_2$ = 0. (44)

The equation πe=1 in (40) gives $\sum_{i=0}^{m*-1}\pi_i e$ + π_{m*}(I-R)$^{-1}$e = 1 (45)

Using π_{m*+1} =$\pi_{m*}R$ and equations (42), (43), (44) and (45) the following matrix equations can be seen where $Q'_{A,2.2}$ is given by (48).

$$(\pi_0,\pi_1,\pi_3,\dots\dots\pi_{m*})Q'_{A,2.2}=0 \quad (46)$$

$$(\pi_0,\pi_1,\pi_3,\dots\dots\pi_{m*})\begin{bmatrix} e \\ (I-R)^{-1}e \end{bmatrix}=1 \quad (47)$$

$$Q'_{A,2.2}=\begin{bmatrix} B'_1 & A_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & A_{2,m*} & RA_2+A_{1,m*} \end{bmatrix} \quad (48)$$

Equations (46) and (47) may be used for finding $(\pi_0, \pi_1, \pi_3, \dots \dots \pi_{m*})$. Replacing the first column of the first column- block in the matrix given by (48) by the column vector multiplier in (47) a matrix which is invertible can be obtained. The first row of the inverse matrix gives $(\pi_0, \pi_1, \pi_3, \dots \dots \pi_{m*})$.This together with equation (41) give all the probability vectors for this Sub Case.

2.3. Performance Measures

(1) The probability P(L = r), of the queue length L = r, can be seen as follows. Let n ≥ 0 and j for 0 ≤ m ≤ M-1 be non-negative integers such that r = n M + m. Then it is noted that P (L=r) $=\sum_{i=1}^{k*}\sum_{j=1}^{k_i}\pi\ (n,m,i,j\ )$, where r = M n + m.

(2) P (Queue length is 0) = P (L=0) = $\sum_{i=1}^{k*}\sum_{j=1}^{k_i}\pi\ (0,0,i,j\ )$.

(3)The expected queue level E(L), can be calculated as follows. For Sub Cases (A1) and (A2) it may be seen as follows. Since $\pi\ (n,m,i,j\ )$ = P [L = M n + m, and environment state = i, arrival BMAP phase=j], for n ≥ 0, 0 ≤ m ≤ M-1, 1 ≤ j ≤ k_i and 1 ≤ i ≤ k*, E(L) $=\sum_{n=0}^{\infty}\sum_{m=0}^{M-1}\sum_{i=1}^{k*}\sum_{j=1}^{k_i}\pi\ (n,m,i,j\ )\ (Mn+m)$ $=\sum_{n=0}^{\infty}\pi_n$. (Mn… Mn, Mn+1… Mn+1, Mn+2…Mn+2… Mn+M-1… Mn+M-1) where in the multiplier vector Mn appears $\sum_{i=1}^{k*}k_i$ times; Mn+1 appears $\sum_{i=1}^{k*}k_i$ times; and so on and finally Mn+M-1appears $\sum_{i=1}^{k*}k_i$ times. So E(L) =M$\sum_{n=0}^{\infty}n\pi_n\ e$ $+\pi_0(\ I-R)^{-1}\xi$. Here ξ is a (M$\sum_{i=1}^{k*}k_i$) x1 type column vector ξ=$(0,\dots 0,1,\dots,1,2,\dots,2,\dots,M-1,\dots,M-1)'$ where 0,1, 2,…M-1 appear $\sum_{i=1}^{k*}k_i$ times in order.This gives E (L) = $\pi_0(\ I-R)^{-1}\xi+M\pi_0(I-R\ )^{-2}Re$. (49)
For Sub Case (A3), E(L) $=\sum_{n=0}^{\infty}\sum_{m=0}^{M-1}\sum_{i=1}^{k*}\sum_{j=1}^{k_i}\pi\ (n,m,i,j\ )\ (Mn+m)$ = M$\sum_{n=0}^{\infty}n\pi_n\ e$ + $\sum_{n=0}^{\infty}\pi_n\ \xi$ = M$\sum_{n=0}^{\infty}n\pi_n\ e$+$\sum_{i=0}^{m*-1}\pi_i$ξ + π_{m*}(I-R)$^{-1}$ξ. Letting the generating function of probability vector Φ(s) = $\sum_{i=0}^{\infty}\pi_i s^i$, it can be seen, Φ(s) = $\sum_{i=0}^{m*-1}\pi_i s^i$ $+\pi_{m*}\ s^{m*}$(I-Rs)$^{-1}$ and $\sum_{n=0}^{\infty}n\pi_n\ e$ = Φ'(1)e = $\sum_{i=0}^{m*-1}i\pi_i\ e$+$\pi_{m*}$m*(I-R)$^{-1}$e + π_{m*}(I-R)$^{-2}$Re. Using this, it is noted that E(L) = M [$\sum_{i=0}^{m*-1}i\pi_i\ e$ + π_{m*}m*(I-R)$^{-1}$e + π_{m*}(I-R)$^{-2}$ Re] +$\sum_{i=0}^{m*-1}\pi_i$ξ + π_{m*}(I-R)$^{-1}$ξ (50)

(4) Variance of queue level can be seen using Var (L) = E (L^2) – E(L)^2. Let η be column vector η=$[0,..0,1^2,\dots 1^2\ 2^2,..2^2,\dots\ (M-1)^2,\dots\ (M-1)^2]'$ of type (M$\sum_{i=1}^{k*}k_i$) x1 where 0,1, 2,…M-1 appear $\sum_{i=1}^{k*}k_i$ times in order. Then it can be seen that the second moment, for Sub Cases (A1) and (A2) E (L^2) = $\sum_{n=0}^{\infty}\sum_{m=0}^{M-1}\sum_{i=1}^{k*}\sum_{j=1}^{k_i}\pi\ (n,m,i,j\ )\ [Mn+m]^2$ $=M^2[\sum_{n=1}^{\infty}n(n-1)\pi_n\ e+\sum_{n=0}^{\infty}n\pi_n\ e]+\sum_{n=0}^{\infty}\pi_n\eta$ + 2M$\sum_{n=0}^{\infty}n\ \pi_n\ \xi$.
So, E(L^2)=$M^2[\pi_0(I-R)^{-3}2R^2\ e+\pi_0(I-R)^{-2}Re]$+$\pi_0(I-R)^{-1}\eta$ + 2M $\pi_0(I-R)^{-2}R\xi$ (51)
Using (49) and (51) the variance can be written for Sub Cases (A1) and (A2). For the Sub Case (A3) the second moment can be seen as follows.
E (L^2) = $\sum_{n=0}^{\infty}\sum_{m=0}^{M-1}\sum_{i=1}^{k*}\sum_{j=1}^{k_i}\pi\ (n,m,i,j\ )\ [Mn+m]^2$ $=M^2[\sum_{n=1}^{\infty}n(n-1)\pi_n\ e+\sum_{n=0}^{\infty}n\pi_n\ e]+\sum_{n=0}^{\infty}\pi_n\eta$ + 2M$\sum_{n=0}^{\infty}n\ \pi_n\ \xi$ = M^2[Φ''(1)e +$\sum_{i=0}^{m*-1}i\pi_i\ e$+$\pi_{m*}$m*(I-R)$^{-1}$e + π_{m*}(I-R)$^{-2}$ Re] + $\sum_{i=0}^{m*-1}\pi_i$η + π_{m*}(I-R)$^{-1}$η + 2M [$\sum_{i=0}^{m*-1}i\pi_i\ \xi$+$\pi_{m*}$m*(I-R)$^{-1}$ξ + π_{m*}(I-R)$^{-2}$ R ξ]. This gives E (L^2) = M^2[$\sum_{i=1}^{m*-1}i(i-1)\pi_i\ e$ + m*(m*-1)π_{m*} $(I-R)^{-1}e$ +2m*π_{m*} (I-R)$^{-2}$Re +2π_{m*}(I-R)$^{-3}$ R^2 e +$\sum_{i=0}^{m*-1}i\pi_i\ e$+$\pi_{m*}$m*(I-R)$^{-1}$e + π_{m*}(I-R)$^{-2}$ Re] + $\sum_{i=0}^{m*-1}\pi_i$η + π_{m*}(I-R)$^{-1}$η +2M [$\sum_{i=0}^{m*-1}i\pi_i\ \xi$+$\pi_{m*}$m*(I-R)$^{-1}$ξ + π_{m*}(I-R)$^{-2}$ R ξ]. (52)
(52) Using (50) and (52) the variance can be written for Sub Case (A3).

III. MODEL (B). MAXIMUM ARRIVAL SIZE M LESS THAN MAXIMUM SERVICE SIZE N

In this Model (B) the dual case of Model (A), namely the case, M < N is treated. Here the partitioning matrices are of order N$\sum_{i=1}^{k*}k_i$ and the customers are considered as members of N blocks. M plays no role in the partition where as it played the major role in Model (A). Two Sub Cases namely (B1) N ≥ C and (B2) C > N come up in the Model (B). (When M =N and for various values of C greater than them, or less than them or equal to them, both Models (A) and (B) are applicable and one can use any one of them.) The assumption (vi) of Model (A) is modified without changing others.

3.1Assumption.

(vi) The maximum batch arrival size of all BMAPs', M= ma$x_{1\leq i\leq k*}M_i$ is greater than the maximum service size N=ma$x_{1\leq i\leq k*}N_i$.

3.2.Analysis

Since this model is dual, the analysis is similar to that of Model (A). The differences are noted below. The state space of the chain is as follows defined in a similar way presented for Model (A). X (t) = {(n, m, i, j): for 0 ≤ m ≤ N-1, for 1 ≤ i ≤ k*, for 1 ≤ j ≤ k_i and 0 ≤ n < ∞}. (53) The chain is in the state (n, m, i, j) when the number of customers in the queue is, n N + m, the environment state is i and the BMAP arrival phase is j for 0 ≤ m ≤ N-1, for 1 ≤ i ≤ k*,for 1 ≤ j ≤ k_i and 0 ≤ n < ∞. When the customers in the system is r then r is identified with (n, m) where r on division by N gives n as the quotient and m as the remainder.

3.2.1 Sub Case: (B1) N ≥ C

The infinitesimal generator $Q_{B,3.1}$ of the Sub Case (B1) of Model (B) has the same block partitioned structure given in (4) for the Sub Cases (A1) and (A2) of Model (A) but the inner matrices are of different orders and elements.

$$Q_{B,3.1}=\begin{bmatrix} B"_1 & A"_0 & 0 & 0 & . & . & . & \cdots \\ A"_2 & A"_1 & A"_0 & 0 & . & . & . & \cdots \\ 0 & A"_2 & A"_1 & A"_0 & 0 & . & . & \cdots \\ 0 & 0 & A"_2 & A"_1 & A"_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A"_2 & A"_1 & A"_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (54)$$

In (54) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots. \underline{n}, \ldots$. Here the vector $\underline{n}$ is of type 1 x N $\sum_{i=1}^{k*} k_i$ and $\underline{n}$ = ((n, 0, 1, 1),(n, 0, 1, 2)…(n, 0, 1, k_1), (n, 0, 2, 1),(n, 0, 2, 2)…(n, 0, 2, k_2),…,(n, 0, k*, 1),(n, 0, k*, 2)…(n, 0, k*, k_{k*}), (n, 1, 1, 1),(n, 1, 1, 2)…(n, 1, 1, k_1), (n, 1, 2, 1),(n, 1, 2, 2)…(n, 1, 2, k_2),…,(n, 1, k*, 1),(n, 1, k*, 2)…(n, 1, k*, k_{k*}),…, (n, N-1, 1, 1),(n, N-1, 1, 2)…(n, N-1, 1, k_1), (n, N-1, 2, 1),(n, N-1, 2, 2)…(n, N-1, 2, k_2),…,(n, N-1, k*, 1),(n, N-1, k*, 2)…(n, N-1, k*, k_{k*})) for n ≥ 0.

The matrices, B''_1, A''_0 , A''_1 *and* A''_2 are all of order N $\sum_{i=1}^{k*} k_i$. The matrices B''_1 *and* A''_1 have negative diagonal elements and their off diagonal elements are non- negative. The matrices A''_0 *and* A''_2 have nonnegative elements. They are all given below. Using the same matrices presented in model (A), for Ω, Λ_j, U_j, V_j , U, Ω' and $Q'_{1,j}$ in (6), (9), (10), (14) to (17) the partitioning matrices are defined below.

$$A'_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 \end{bmatrix} \quad (55)$$

$$A''_2 \begin{bmatrix} CU_N & CU_{N-1} & CU_{N-2} & \cdots & CU_2 & CU_1 \\ 0 & CU_N & CU_{N-1} & \cdots & CU_3 & CU_2 \\ 0 & 0 & CU_N & \cdots & CU_4 & CU_3 \\ 0 & 0 & 0 & \ddots & CU_5 & CU_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & CU_{N-1} & CU_{N-2} \\ 0 & 0 & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & CU_N \end{bmatrix} \quad (56)$$

$$A''_1 = \begin{bmatrix} \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0 \\ CU_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ CU_2 & CU_1 & \Omega & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-3} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M \\ CU_{N-M} & CU_{N-M-1} & CU_{N-M-2} & \cdots & CU_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ CU_{N-M+1} & CU_{N-M} & CU_{N-M-1} & \cdots & CU_2 & CU_1 & \Omega & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{N-2} & CU_{N-3} & CU_{N-4} & \cdots & CU_{N-M-2} & CU_{N-M-3} & CU_{N-M-2} & \cdots & \Omega & \Lambda_1 \\ CU_{N-1} & CU_{N-2} & CU_{N-3} & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-1} & \cdots & CU_1 & \Omega \end{bmatrix} (57)$$

ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 05, Issue 03 (March. 2015), ||V1|| PP 33-47

$$B''_1 = \begin{bmatrix} Q'_{1,0} & \Lambda_1 & \Lambda_2 & \cdots & . & \cdots & \Lambda_M & 0 & 0 & \cdots & . & . & \cdots & 0 & 0 \\ U & Q'_{1,1} & \Lambda_1 & \cdots & . & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & . & . & \cdots & 0 & 0 \\ 2V_1 & 2U_1 & Q'_{1,2} & \cdots & . & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & . & . & \cdots & 0 & 0 \\ 3V_2 & 3U_2 & 3U_1 & \cdots & . & \cdots & \vdots & \vdots & \vdots & \cdots & . & . & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CV_{C-1} & CU_{C-1} & CU_{C-2} & \cdots & \Omega & \cdots & . & . & . & \cdots & \Lambda_M & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CV_{N-M-2} & CU_{N-M-2} & CU_{N-M-3} & \cdots & . & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & . & . & \cdots & \Lambda_{M-1} & \Lambda_M \\ CV_{N-M-1} & CU_{N-M-1} & CU_{N-M-2} & \cdots & . & \cdots & CU_1 & \Omega & \Lambda_1 & \cdots & . & . & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ \vdots & \vdots & \vdots & \ddots & . & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CV_{N-3} & CU_{N-3} & CU_{N-4} & \cdots & . & \cdots & CU_{N-M-2} & CU_{N-M-3} & CU_{N-M-2} & \cdots & . & . & \cdots & \Omega & \Lambda_1 \\ CV_{N-2} & CU_{N-2} & CU_{N-3} & \cdots & . & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-1} & \cdots & . & . & \cdots & CU_1 & \Omega \end{bmatrix} \quad (58)$$

In (58) the case N > C has been presented. When C=N, V_j and U_j in B''_1 do not get C as multiplier in (58) and C appears as a multiplier of U_j in A''_2 and A''_1 in (56) and (57). The multiplier of matrices U_j *and* V_j concerning the services increases by one in each row block from third row block as the row number increases by one, up to the row C+1 and it remains C in row blocks after that as given above.

$$Q''_B = \begin{bmatrix} \Omega + CU_N & \Lambda_1 + CU_{N-1} & \cdots & \Lambda_{M-1} + CU_{N-M+1} & \Lambda_M + CU_{N-M} & CU_{N-M-1} & \cdots & CU_2 & CU_1 \\ CU_1 & \Omega + CU_N & \cdots & \Lambda_{M-2} + CU_{N-M+2} & \Lambda_{M-1} + CU_{N-M+1} & \Lambda_M + CU_{N-M} & \cdots & CU_3 & CU_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ CU_{N-M-2} & CU_{N-M-3} & \cdots & \Omega + CU_N & \Lambda_1 + cU_{N-1} & \Lambda_2 + CU_{N-2} & \cdots & \Lambda_M + CU_{N-M} & CU_{N-M-1} \\ CU_{N-M-1} & CU_{N-M-2} & \cdots & CU_1 & \Omega + CU_N & \Lambda_1 + CU_{N-1} & \cdots & \Lambda_{M-1} + CU_{N-M+1} & \Lambda_M + CU_{N-M} \\ \Lambda_M + CU_{N-M} & CU_{N-M-1} & \cdots & CU_2 & CU_1 & \Omega + CU_N & \cdots & \Lambda_{M-2} + CU_{N-M+2} & \Lambda_{M-1} + CU_{N-M+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 + CU_{N-2} & \Lambda_3 + CU_{N-3} & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-3} & \cdots & \Omega + CU_N & \Lambda_1 + CU_{N-1} \\ \Lambda_1 + CU_{N-1} & \Lambda_2 + CU_{N-2} & \cdots & \Lambda_M + CU_{N-M} & CU_{N-M-1} & CU_{N-M-2} & \cdots & CU_1 & \Omega + CU_N \end{bmatrix} \quad (59)$$

The basic generator (59) which is concerned with only the arrival and service is $Q''_B = A''_0 + A''_1 + A''_2$. This is also block circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is w'= $\left(\frac{w}{N}, \frac{w}{N}, \frac{w}{N}, \ldots\ldots, \frac{w}{N}\right)$ where w is as seen in Model (A), where w= $(\phi_1\varphi_1, \phi_2\varphi_2, \ldots, \phi_{k*}\varphi_{k*})$ and the stability condition remains the same as in Model (A). Following the arguments given for Sub Cases (A1) and (A2) of Model (A), one can find the stationary probability vector for Sub Case (B1) of Model (B) also in matrix geometric form. All performance measures in section 2.3 including the expectation of customers waiting for service and its variance for Sub Cases (A1) and (A2) of Model (A) are valid for Sub Case (B1) of Model (B) with M is replaced by N. It can also be seen that when N = C the system admits Matrix Geometric solution as in Model (A).

3.2.2Sub Case: (B2) C > N

The infinitesimal generator $Q_{B,3.2}$ of the Sub Case (B2) of Model (B) has the same block partitioned structure given in (34) for Sub Case (A3) of Model (A) but the inner matrices are of different orders and elements. When C > N > M, the BMAP/M/C bulk queue admits a modified matrix geometric solution as follows. The chain X (t) describing this Sub Case (B2), can be defined as in the Sub Case (B1). It has the infinitesimal generator $Q_{B,3.2}$ of infinite order which can be presented in block partitioned form given below. When C > N, let C = m* N + n* where m* is positive integer and n* is nonnegative integer with 0 ≤ n* ≤ N-1.

$$Q_{B,3.2} = \begin{bmatrix} B'''_1 & A''_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ A''_{2,1} & A''_{1,1} & A''_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & A''_{2,2} & A''_{1,2} & A''_0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A''_{2,3} & A''_{1,3} & A''_0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & A''_{2,m*} & A''_{1,m*} & A''_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & A''_2 & A''_1 & A''_0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & A''_2 & A''_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots \end{bmatrix} \quad (60)$$

In (60) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots.\underline{n}, \ldots$. Here the vector $\underline{n}$ is of type 1 x N $\sum_{i=1}^{k*} k_i$ and $\underline{n}$ = ((n, 0, 1, 1),(n, 0, 1, 2)…(n, 0, 1, k_1), (n, 0, 2, 1),(n, 0, 2, 2)…(n, 0, 2, k_2),…,(n, 0, k*, 1),(n, 0, k*, 2)…(n, 0, k*, k_{k*}), (n, 1, 1, 1),(n, 1, 1, 2)…(n, 1, 1, k_1), (n, 1, 2, 1),(n, 1, 2, 2)…(n, 1, 2,

k_2),…,(n, 1, k*, 1),(n, 1, k*, 2)…(n, 1, k*, k_{k*}),…, (n, N-1, 1, 1),(n, N-1, 1, 2)…(n, N-1, 1, k_1), (n, N-1, 2, 1),(n, N-1, 2, 2)…(n, N-1, 2, k_2),…,(n, N-1, k*, 1),(n, N-1, k*, 2)…(n, N-1, k*, k_{k*})) for n ≥ 0.

The matrices B'''_1, A''_{1j} for 1 ≤ j ≤ m* and A''_1 have negative diagonal elements, they are of order N $\sum_{i=1}^{k*} k_i$ and their off diagonal elements are non- negative. The matrices $A''_0, A''_{2,j}$ *and* A''_2 for 1 ≤ j ≤ m* have nonnegative elements and are of order N $\sum_{i=1}^{k*} k_i$ and the matrices A''_0, A''_1 *and* A''_2 are same as defined earlier for Sub Case (B1) in equations (55), (56) and (57). Since C > N the number of servers in the system s equals the number of customers in the system L up to customer length becomes C= m* N + n*. Once number of customers L ≥ C, the number of servers in the system remains C. When the number of customers becomes less than C, the number of servers again falls and equals the number of customers. Using the same matrices presented in model (A), for Ω, Λ_j, U_j, V_j U, Ω' and $Q'_{1,j}$ in (6), (9), (10), (14) to (17) the partitioning matrices are defined below.

The matrix $A''_{2,j}$ is given for 1 ≤ j < m*-1, as

$$A''_{2,j} = \begin{bmatrix} jNU_N & jNU_{N-1} & \cdots & jNU_2 & jNU_1 \\ 0 & (jN+1)U_N & \cdots & (jN+1)U_3 & (jN+1)U_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (jN+N-2)U_N & (jN+N-2)U_{N-1} \\ 0 & 0 & \cdots & 0 & (jN+N-1)U_N \end{bmatrix} \quad (61)$$

The matrix $A_{2,m*}$ is as follows given in (62) when C = m*N + n* where 0 ≤ n* ≤ N-1.

$$A''_{2,m*} = \begin{bmatrix} (Nm*)U_N & (Nm*)U_{N-1} & \cdots & . & \cdots & (Nm*)U_2 & (Nm*)U_1 \\ 0 & (Nm*+1)U_N & \cdots & . & \cdots & (Nm*+1)U_3 & (Nm*+1)U_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & CU_N & \cdots & CU_{n*+2} & CU_{n*+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & CU_N & CU_{N-1} \\ 0 & 0 & \cdots & 0 & \cdots & 0 & CU_N \end{bmatrix} \quad (62)$$

Let $Q'_{1,j} = \Omega' - jU$ for 0 ≤ j ≤ C and $Q'_{1,C} = \Omega$ as in Sub Cases (A1) and(A2). Then B'''_1, is defined as follows.

$$B'''_1 = \begin{bmatrix} Q'_{1,0} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & \cdots & 0 & 0 \\ U & Q'_{1,1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ 2V_1 & 2U_1 & Q'_{1,2} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 \\ 3V_2 & 3U_2 & 3U_1 & \cdots & . & . & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (N-M-1)V_{N-M-2} & (N-M-1)U_{N-M-2} & (N-M-1)U_{N-M-3} & \cdots & Q'_{1,N-M-1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (N-2)V_{N-3} & (N-2)U_{N-3} & (N-2)U_{N-4} & \cdots & (N-2)U_{N-M-2} & (N-2)U_{N-M-3} & \cdots & Q'_{1,N-2} & \Lambda_1 \\ (N-1)V_{N-2} & (N-1)U_{N-2} & (N-1)U_{N-3} & \cdots & (N-1)U_{N-M-1} & (N-1)U_{N-M-2} & \cdots & (N-1)U_1 & Q'_{1,N-1} \end{bmatrix} \quad (63)$$

The matrix $A''_{1,j}$ for 1 ≤ j ≤ m*-1 is as follows.

$$A''_{1,j} = \begin{bmatrix} Q'_{1,jN} & \Lambda_1 & \cdots & \Lambda_M & 0 & \cdots & 0 & 0 \\ (jN+1)U_1 & Q'_{1,jN+1} & \cdots & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ (jN+2)U_2 & (jN+2)U_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (jN+N-M-1)U_{N-M-1} & (jN+N-M-1)U_{N-M-2} & \cdots & Q'_{1,jN+N-M-1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (jN+N-2)U_{N-2} & (jN+N-2)U_{N-3} & \cdots & (jN+N-2)U_{N-2-M} & (jN+N-2)U_{N-3-M} & \cdots & Q'_{1,jN+N-2} & \Lambda_1 \\ (jN+N-1)U_{N-1} & (jN+N-1)U_{N-2} & \cdots & (jN+N-1)U_{N-1-M} & (jM+M-1)U_{N-2-M} & \cdots & (jM+M-1)U_1 & Q'_{1,jN+N-1} \end{bmatrix} \quad (64)$$

The matrix $A''_{1,m*}$ is in (65) when C = m*N + n* and 0 ≤ n* ≤ N-1. From row block n*+1, the multiplier of U_j is C.

$$A''_{1,m*} = \begin{bmatrix} Q'_{1,Nm*} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0 \\ (Nm*+1)U_1 & Q'_{1,Nm*+1} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ (Nm*+2)U_2 & (Nm*+2)U_1 & Q'_{1,Nm*+2} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{n*} & CU_{n*-1} & CU_{n*-2} & \cdots & Q'_{1,C} & \Lambda_1 & \Lambda_2 & \cdots & . & . \\ CU_{n*+1} & CU_{n*} & CU_{n*-1} & \cdots & CU_1 & Q'_{1C} & \Lambda_1 & \cdots & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CU_{N-2} & CU_{N-3} & CU_{N-4} & \cdots & CU_{N-M-2} & CU_{N-M-3} & CU_{N-M-2} & \cdots & Q'_{1C} & \Lambda_1 \\ CU_{N-1} & CU_{N-2} & CU_{N-3} & \cdots & CU_{N-M-1} & CU_{N-M-2} & CU_{N-M-1} & \cdots & CU_1 & Q'_{1C} \end{bmatrix} \quad (65)$$

The basic generator for this model is also same as (59) which is concerned with only the arrival and service. $Q''_B = A''_0 + A''_1 + A''_2$. This is also block circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is $w' = \left(\frac{w}{N}, \frac{w}{N}, \frac{w}{N}, \ldots, \frac{w}{N}\right)$, where w = $(\phi_1\varphi_1, \phi_2\varphi_2, \ldots, \phi_{k*}\varphi_{k*})$ and the stability condition remains the same. Following the arguments given for Sub Case (A3) in section 2.2.2 of Model (A), one can find the stationary probability vector for Sub Case (B2) of Model (B) also in modified matrix geometric form. All the performance measures given in section 2.3 including the expectation of customers waiting for service and its variance for Sub Case (A3) are valid for Sub Case (B2) of Model (B) except M is replaced by N.

IV. NUMERICAL ILLUSTRATION

For the BMAP/M/C bulk models, the varying environment is considered to be governed by the Matrix $Q_1 = \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix}$. Nine examples three for each are studied for the cases M = N =3; M = 3, N= 2 and M = 2, N = 3 with the number of servers in each case as C = 2, 3 and 4. Matrix geometric results are seen for C = 2 and C = 3 ≤ M or N. Modified Matrix Geometric results are seen when C = 4 > M and N.

The service time parameters of exponential distributions are respectively fixed in the two environments E1 and E2 as $\mu_1 = 5 \; and \; \mu_2 = .5$ for single server respectively.

For the case M=3, BMAP, the batch Markovian arrival process for E1 is given by $D_0^1 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$, $D_1^1 = \begin{bmatrix} .2 & .3 \\ .32 & .48 \end{bmatrix}$, $D_2^1 = \begin{bmatrix} .12 & .18 \\ .08 & .12 \end{bmatrix}$, $D_3^1 = \begin{bmatrix} .08 & .12 \\ 0 & 0 \end{bmatrix}$ and BMAP for the environment E2 is given by $D_0^2 = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix}$, $D_1^2 = \begin{bmatrix} .72 & .48 \\ 1.62 & 1.08 \end{bmatrix}$, $D_2^2 = \begin{bmatrix} .48 & .32 \\ .18 & .12 \end{bmatrix}$, $D_3^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

For the case M=2, D_i^1 for i=0 and i=1 and D_i^2 for i = 0, 1, 2, 3 are as given above for the case M = 3 but it is assumed that $D_2^1 = \begin{bmatrix} .2 & .3 \\ .08 & .12 \end{bmatrix}$ and $D_3^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The bulk size service probabilities are given in table 1for the case when M = N = 3 for the two environment.

For the case M =3, N=2 the probabilities of bulk service size 2 in E1 is fixed as .5 and of bulk size 3 in E1 is fixed as 0; and other probabilities are unchanged.

Table 1: Service probabilities

Environment 1	P(size 1)	P(size 2)	P(size 3)	Environment 2	P(size 1)	P(size 2)	P(size 3)
Service	.5	.3	.2	Service	.8	.2	0

Thirty iterations are performed for all the models to iterate the rate Matrix R and the norms of convergence are recorded. Queue length probabilities and block size probabilities are calculated. Expected queue length and Standard deviation are presented. They show significant variations when M, N and C are changed. The probabilities of queue lengths and block sizes are presented in figures 1 and 2 for all the nine examples.

Table2: Results Obtained For Six Matrix Geometric Models with Servers C=2, 3 and Three Modified Matrix Geometric Models with Servers C=4.

Servers	C=2	C=2	C=2	C=3	C=3	C=3	C=4	C=4	C=4
Bulk sizes	M=3=N	M=3,N=2	M=2,N=3	M=3=N	M=3,N=2	M=2,N=3	M=3=N	M=3,N=2	M=2,N=3
P(L=0)	0.11704754	0.09261693	0.11917855	0.19277723	0.17666039	0.19396124	0.22512063	0.2034006	0.21803202
P(L=1)	0.09471811	0.07956358	0.09633196	0.15959817	0.15559931	0.16039981	0.19034574	0.18030897	0.18162383
P(L=2)	0.07960545	0.06780569	0.08129609	0.14005869	0.13775703	0.14131685	0.17357565	0.16232437	0.16281021
P(L=3)	0.07205532	0.06272839	0.07260304	0.10437229	0.10383676	0.10417925	0.11225346	0.12649453	0.12451202
P(L=4)	0.06447664	0.05733307	0.06494065	0.08344299	0.08403562	0.08326405	0.08280449	0.08827898	0.08635215
π0e	0.2913711	0.2399862	0.2968066	0.49243409	0.47001673	0.4956779	0.58904202	0.54603394	0.56246606
π1e	0.19445266	0.1726533	0.19578838	0.25380206	0.2551476	0.2531594	0.25460537	0.27958837	0.27367983
π2e	0.1408797	0.13323249	0.14114802	0.12692234	0.13234822	0.1261417	0.09700232	0.10667238	0.10183472
π3e	0.10227831	0.10300958	0.10188375	0.06343794	0.06861222	0.06278691	0.03681867	0.04141741	0.03854677
π(n>3)e	0.27101824	0.35111843	0.26437325	0.06340357	0.07387523	0.06223409	0.02253161	0.0262879	0.02347262
Norm	9.05E-05	0.00022995	8.4444E-05	2.8292E-07	8.0941E-07	2.6120E-07	2.7034E-09	6.9165E-09	2.5168E-09
Arri Rate	0.91545894	0.91545894	0.90879227	0.91545894	0.91545894	0.90879227	0.91545894	0.91545894	0.90879227
Serv rate	1.27777778	1.16666667	1.27777778	1.91666667	1.75	1.91666667	2.55555556	2.33333333	2.55555556
E(L)	8.66769282	10.9797998	8.49277631	3.91495972	4.18689538	3.88357328	2.96706773	3.07824887	2.95184216
Std dev(L)	9.34497897	11.6425663	9.17632301	4.31785421	4.56077391	4.289501	3.15889414	3.23338421	3.1466112

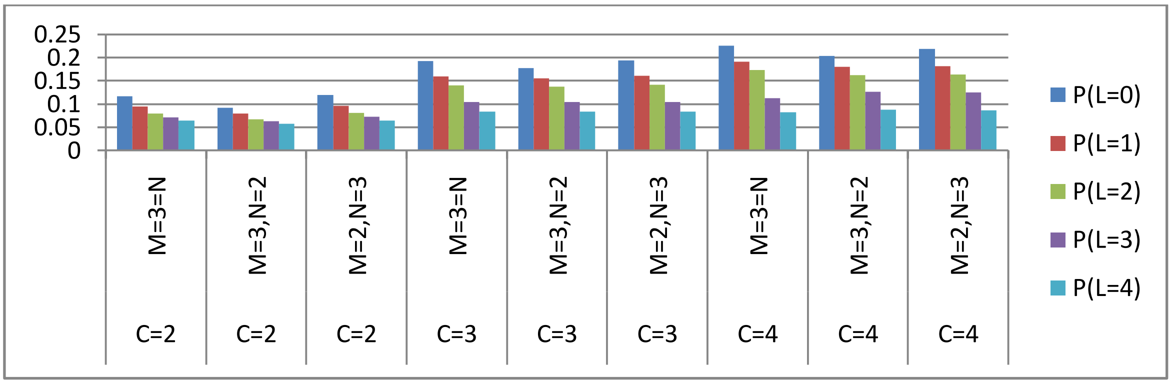

Figure 1: Probabilities of Queue lengths

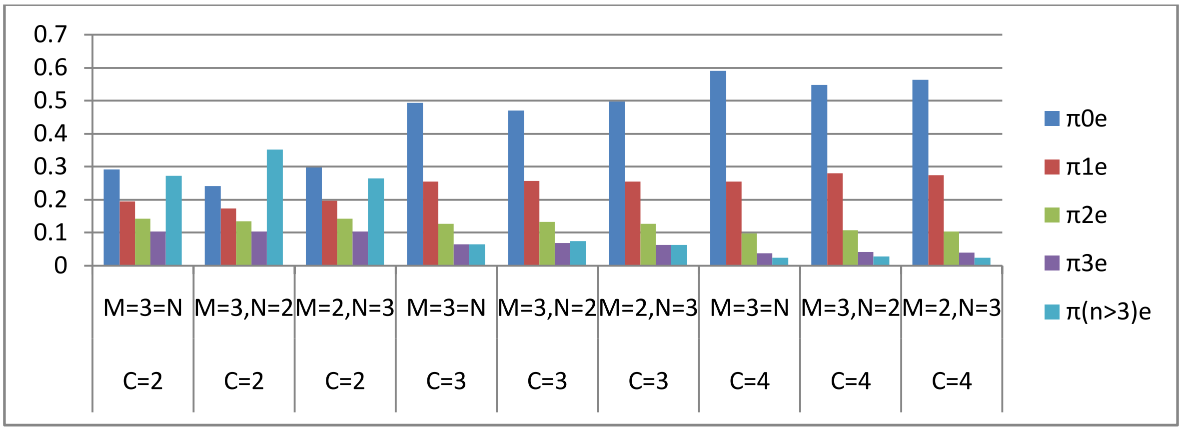

Figure 2: Probabilities of Block Sizes

V. CONCLUSION

Two BMAP/M/C bulk service queues and their sub cases with randomly varying environments have been studied. The environment changes the batch Markovian arrival processes, the service rates, and the probabilities of bulk services. Matrix geometric (modified matrix geometric) results have been obtained by suitably partitioning the infinitesimal generator by grouping of customers, environments, BMAP and PH phases together respectively when the number of servers is not greater than (greater than) the maximum of the maximum arrival and maximum service sizes. The basic system generators of the queues are block circulant matrices which are explicitly presenting the stability condition in standard form. Numerical results for various bulk queue models are presented and discussed. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited. The decrease in arrival rates (so also increase in service rates) makes the convergence of R matrix faster which can be seen in the decrease of norm values. Bulk BMAP/PH/C queue with randomly varying environments causing changes in sizes of the PH phases may produce further results if studied since BMAP/PH/C queue is a most general form almost equivalent to G/G/C queue.

VI. ACKNOWLEDGEMENT

The fourth author thanks ANSYS Inc., USA, for providing facilities. The contents of the article published are the responsibilities of the authors.

REFERENCES

[1]. Rama Ganesan, Ramshankar.R and Ramanarayanan. R (2014) M/M/1 Bulk Arrival and Bulk Service Queue with Randomly Varying Environment, IOSR-JM, Vol.10,Issue 6, Ver III, Nov-Dec, 58-66

[2]. Sandhya.R, Sundar.V, Rama.G, Ramshankar.R and Ramanarayanan.R, (2015) M/M//C Bulk Arrival And Bulk Service Queue With Randomly Varying Environment,IOSR-JEN,Vol.05,Issue02,||V1||,pp.13-26.

[3]. Ramshankar.R, Rama Ganesan and Ramanarayanan.R (2015) PH/PH/1 Bulk Arrival and Bulk Service Queue ,IJCA, Vol. 109,.3,27-33

[4]. Ramshankar.r, Rama.G, Sandhya.R, Sundar.V, Ramanarayanan.R , (2015), PH/PH/1 Bulk Arrival and Bulk Service Queue with Randomly Varying Environment, IOSRJEN, Vol.05,Issue.02,||V4|| pp.01-12.

[5]. Bini.D, Latouche.G and Meini.B (2005). Numerical methods for structured Markov chains, Oxford Univ. Press, Oxford.

[6]. Chakravarthy.S.R and Neuts. M.F.(2014). Analysis of multi-server queue model with MAP arrivals of customers, SMPT, Vol 43,79-95,

[7]. Gaver, D., Jacobs, P., Latouche, G, (1984). Finite birth-and-death models in randomly changing environments. AAP. 16, 715–731

[8]. Latouche.G, and Ramaswami.V, (1998). Introduction to Matrix Analytic Methods in Stochastic Modeling, SIAM. Philadelphia.

[9]. Neuts.M.F.(1981).Matrix-Geometric Solutions in Stochastic Models: An algorithmic Approach, The Johns Hopkins Press, Baltimore

[10]. Lucantony.D.M. (1993), The BMAP/G/1 Queue: A tutorial, Models and Techniques for Performance Evaluation of Computer and Communication Systems, L. Donatiello and R. Nelson Editors, Springer Verlag, pp 330-358.

[11]. Cordeiro.J.D and Kharoufch. J.P, (2011) Batch Markovian Arrival Processes, www.dtic.mil/get-tr-doc/pdf?AD=ADA536697 & origin=publication Detail.

[12]. Usha.K,(1981), Contribution to The Study of Stochastic Models in Reliability and Queues, PhD Thesis, Annamlai University,India.

[13]. Usha.K,(1984), The PH/M/C Queue with Varying Environment, Zastosowania Matematyki Applications Mathematicae, XVIII,2 pp.169-175.

[14]. Qi-Ming-He,(2014), Fundamentals-of-Matrix-Analytic-Methods,Springer.